In milliamps, this becomes $\mathbf{i} = 3.6\angle 35.3^\circ$, which is the result obtained earlier, in the box on p. 54.

This example illustrates that when we write impedances as complex numbers we handle them by applying the usual rules for resistances in series or in parallel. There is no need to draw a phasor diagram because phase is taken into account by the complex representation of reactances.

Test yourself

1. A circuit has a voltage generator, a 47 kΩ resistor and a 220 nF capacitor in series. The generator produces a pd signal, $v = 8 \sin 100\pi t$. Use the phasor technique to find the pd signals across the resistor and the capacitor, with phase relative to the generator signal.
2. A circuit has a voltage generator, a 120 Ω resistor and a 10 mH inductor in series. The generator produces a signal, $v = 4.5 \sin 6000\pi t$. Use the phasor technique to find the pd signals across the resistor and the inductor, with phase relative to the generator signal.
3. In the circuit of question 1 the pd signal is changed to $v = 6 \sin 200\pi t$. Express the impedances as complex values and use these to find the total circuit impedance, the current signal and the pd signal across the capacitor. Express the results as phasors in rectangular form, phasors in polar form, and as equations for current or pd.
4. In the circuit of question 2 the pd signal is changed to $v = 5 \sin 4000\pi t$. Express the impedances as complex values and use these to find the total circuit impedance, the current signal and the pd signal across the inductor. Express the results as phasors in rectangular form, phasors in polar form, and as equations for current or pd.

L.P $H(j\omega) = \frac{V_o}{V_{in}} = \frac{1}{1 + j\omega RC}$

H.P $H(j\omega) = \frac{v_o}{v_{in}} = \frac{j\omega RC}{1 + j\omega RC}$

L.P $\phi = -\tan^{-1} \omega RC$

H.P $= \pi/2 - \tan^{-1} \omega RC$

$Q = \frac{f_r}{B.W}$

understanding Electronic filters

5 Passive filters

The function of a filter is to allow signals of a given band of frequencies to pass, while obstructing or reducing the amplitude of others. In this context we are referring to sinusoidal signals, since signals of other forms are considered to be a mixture of sinusoids of different frequencies and different amplitudes. Figures 2.12 and 2.13 show the composition of typical sawtooth and triangular waves. If these signals are passed through a filter, some of the frequencies may be stopped altogether from passing while others may be attenuated (reduced in amplitude). The frequency spectrum has a markedly different appearance after the signals have passed through a filter.

Before considering the effects of filters on triangular and other composite waveforms, we will look at what happens to a pure sine wave of a single frequency. The essentials of this have already been described in earlier chapters but now we represent the facts in the context of a filter circuit. This chapter deals with passive filters, and the simplest type of filter is built from two passive components, a resistor and a capacitor. The circuit of the filter illustrated in Fig. 5.1 is exactly the same as that of Fig. 1.5. The pd source is drawn in dashed lines because it is not part of the filter. It could be any device or circuit that produces a varying pd, $\mathbf{v}_{in}$. It might be a microphone, a photo-cell, a circuit based on a thermistor, a radio antenna, a pair of electrodes in an electrocardiograph, or one of thousands of other sources of varying pd. For the purpose of this discussion, it is the source of a sine wave signal of a single frequency.

The other special feature of this circuit is that there are connections on either side of the capacitor to convey a signal $\mathbf{v}_{out}$ to an external circuit. It is essential that the following circuit draws as little current as possible from the filter. Otherwise, the action of the filter is degraded. The following circuit must have a high input impedance and we assume in the discussions below that its input impedance is so high as to draw no appreciable current from the filter. It is also assumed that the pd source has a suitably low output impedance so that it is able to supply as much current as the filter can accept at any instant.

The filter of Fig. 5.1 is labelled with complex impedances, as discussed at the end of the last chapter. Only the capacitor has a complex impedance in this circuit. The total impedance of the filter is the sum of the impedances of the resistor and capacitor in series:

$$Z = R - j/\omega C$$

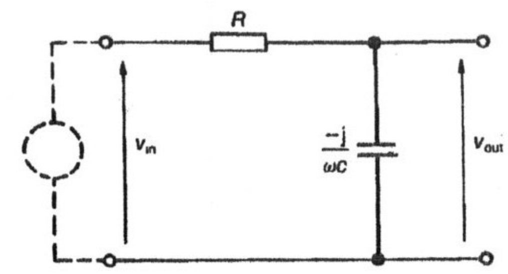

Figure 5.1 *The circuit of Fig. 1.5 redrawn as a low-pass passive filter*

For example, if the resistance is 220 Ω, the capacitance is 1 μF and the frequency is 1 kHz, then $\omega = 2\pi f = 6283$ rad/s and:

$$Z = 220 - j159$$

The impedance is in ohms, as usual. Take $\mathbf{v_{in}}$ and $\mathbf{v_{out}}$ as symbols for the input and output phasors. When $\mathbf{v_{in}}$ is applied to the filter, the resistor and capacitor act as a potential divider. Therefore $\mathbf{v_{out}}$, the pd signal across the capacitor, bears the same proportion to $\mathbf{v_{in}}$ as the impedance of the capacitor bears to the total impedance.

$$\frac{\mathbf{v_{out}}}{\mathbf{v_{in}}} = \frac{-j159}{220 - j159}$$

or

$$\mathbf{v_{out}} = \frac{\mathbf{v_{in}} \times -j159}{220 - j159}$$

If the source signal is $\mathbf{v_{in}} = 2\sin 2000\pi t$, and is taken to be the reference signal with $\phi = 0°$, $\mathbf{v_{in}} = 2 + j0 = 2$, and:

$$\mathbf{v_{out}} = \frac{-j318}{220 - j159} = 0.686 - j0.949$$

The technique for dealing with a complex divisor is explained at the end of Chapter 4. Converting $\mathbf{v_{out}}$ into polar form:

$$\mathbf{v_{out}} = 1.17\underline{/-54.1°}$$

Writing this as an equation for the output signal:

$$\mathbf{v_{out}} = 1.17\sin(2000\pi t - 54.1°)$$

Figure 5.2 shows the curves for $\mathbf{v_{in}}$ and $\mathbf{v_{out}}$. After the initial stage while the capacitor gains charge, it is clear that $\mathbf{v_{out}}$ has the same frequency as $\mathbf{v_{in}}$. Measurements on the graph show that the amplitude of $\mathbf{v_{out}}$ is 1.17 V, and that its phase lag is 54°. The gain of the filter is 1.17/2 = 0.585. It is a characteristic of all passive filters that the output signal has a smaller amplitude than the input signal, in other words, that it has a gain less than unity. This can be seen on the phasor diagram of the filter (Fig. 5.3).

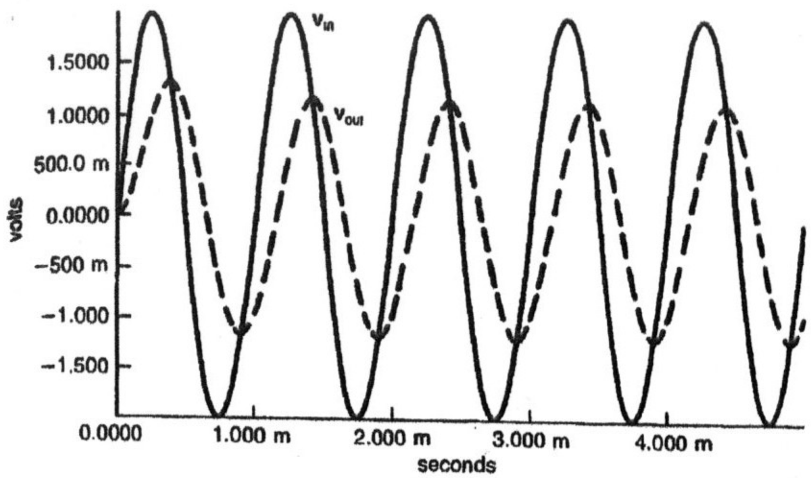

Figure 5.2 *When the frequency is 1 kHz, the output of the filter of Fig. 5.1 has reduced amplitude and appreciable phase lag*

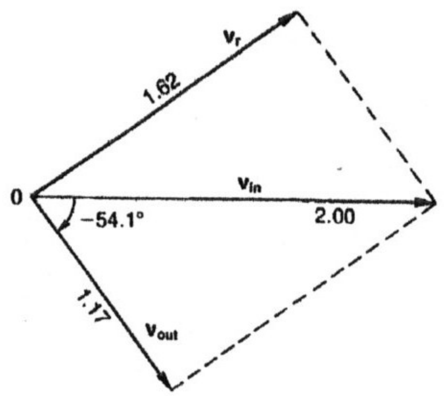

Figure 5.3 *A phasor diagram shows the relationship between the input and output of the filter of Fig. 5.1 when the frequency is 1 kHz*

Keeping up?

1. Given a low-pass filter built from a 470 Ω resistor and a 2.2 μF capacitor, with an input signal $\mathbf{v_{in}} = 4\sin 300\pi t$, calculate the output signal $\mathbf{v_{out}}$ and express it as an equation, in complex (rectangular) form and in polar form.
2. If the capacitor and resistor in Fig. 5.1 are exchanged, so that $\mathbf{v_{out}}$ becomes the pd across the resistor, what kind of filter does this produce? Given the

values listed in question 1, calculate the output signal $\mathbf{v}_{out}$ and express it as an equation, in complex (rectangular) form and in polar form.

Variable impedance

The filter circuit of Fig. 5.1 can be thought of as a potential divider. The capacitor is equivalent to a variable resistor controlled by frequency (Fig. 5.4). The higher the frequency, the lower the impedance of the capacitor and the smaller the amplitude of $\mathbf{v}_{out}$. The action is that of a low-pass filter.

General equation

Having looked at an actual example, we will work through the same problem but without inserting actual component values. For the input signal, let:

$$\mathbf{v}_{in} = V_0 \sin \omega t$$

Then the total circuit impedance is:

$$R - \mathrm{j}/\omega C$$

The corresponding output signal is:

$$\mathbf{v}_{out} = \frac{\mathbf{v}_{in} \times -\mathrm{j}/\omega C}{R - \mathrm{j}/\omega C}$$

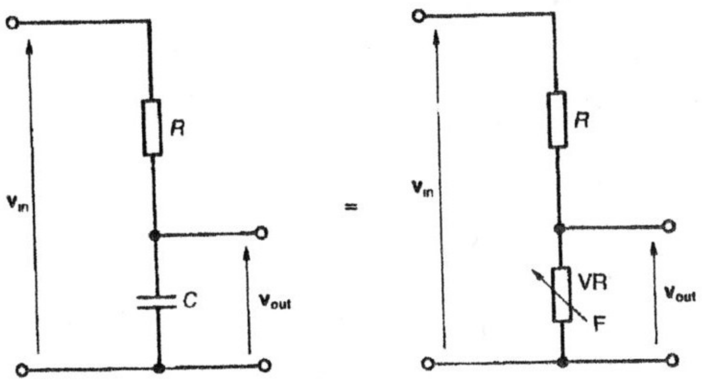

Figure 5.4 *Another way of looking at a low-pass resistor/capacitor filter*

Multiply top and bottom by $\mathrm{j}\omega C$:

$$\mathbf{v}_{out} = \frac{\mathbf{v}_{in}}{\mathrm{j}\omega RC + 1}$$

We leave the equation in this form. If you substitute values of $\mathbf{v}_{in}$, ω, R and C, you can confirm the equation by obtaining the same results as above. In its general form, the equation shows that the output is larger if:

- input is larger
- frequency is smaller
- R is smaller
- C is smaller

Obviously we obtain a larger output with a larger input, otherwise the outgoing signal would not be a replica of the ingoing signal. Making ω smaller increases the impedance of the capacitor and hence the pd across it. Making R smaller increases the current flowing in the circuit and therefore increases the pd across the capacitor. Making C smaller causes larger changes of pd for a given change of charge ($v = q/C$, see equation (6) in Chapter 1).

If input, R and C are held constant, the amplitude of the output depends on ω. The equation shows that the output signal amplitude increases as ω decreases. This is seen in Fig. 5.5 where the results of reworking the numerical example above for four different frequencies are shown side by side. The amplitude of the input signal and the values of R and C are left unchanged. Looking at the diagrams in order of frequency we see that, as frequency decreases, the proportions of the rectangle OPQR change, while the length of the diagonal (source phasor) remains constant. With decreasing frequency the signal across the resistor becomes smaller while that across the capacitor becomes larger. In other words $\mathbf{v}_{out}$ increases. At the same time the output phasor swings round nearer and nearer to the source phasor. It lags less and less far behind the source phasor. Continuing this trend it is possible to imagine the output phasor swinging round to coincide with the source phasor when frequency is very low. At the limit, the amplitude of the output signal becomes equal to that of the input signal and is in phase with it. The signal passes unaffected through the filter. This effect can be recognized in the equation above as ω approaches a limit of 0:

$$\mathbf{v}_{out} = \frac{\mathbf{v}_{in}}{\mathrm{j}0RC + 1} = \frac{\mathbf{v}_{in}}{1} = \mathbf{v}_{in}$$

Since $\mathbf{v}_{in}$ has no imaginary part, neither does $\mathbf{v}_{out}$ and the phase angle is zero. A zero-frequency signal is a constant DC level. If a DC pd is applied to the filter, the capacitor quickly charges to that level. There is no fall in pd across the resistor (we are assuming that the following circuit is drawing no current) so $\mathbf{v}_{out} = \mathbf{v}_{in}$.

At the other extreme, at very high frequency, $\mathbf{v}_{out}$ approaches zero, with −90° phase angle. A diagram to summarize these changes can be plotted by using a circuit simulator.

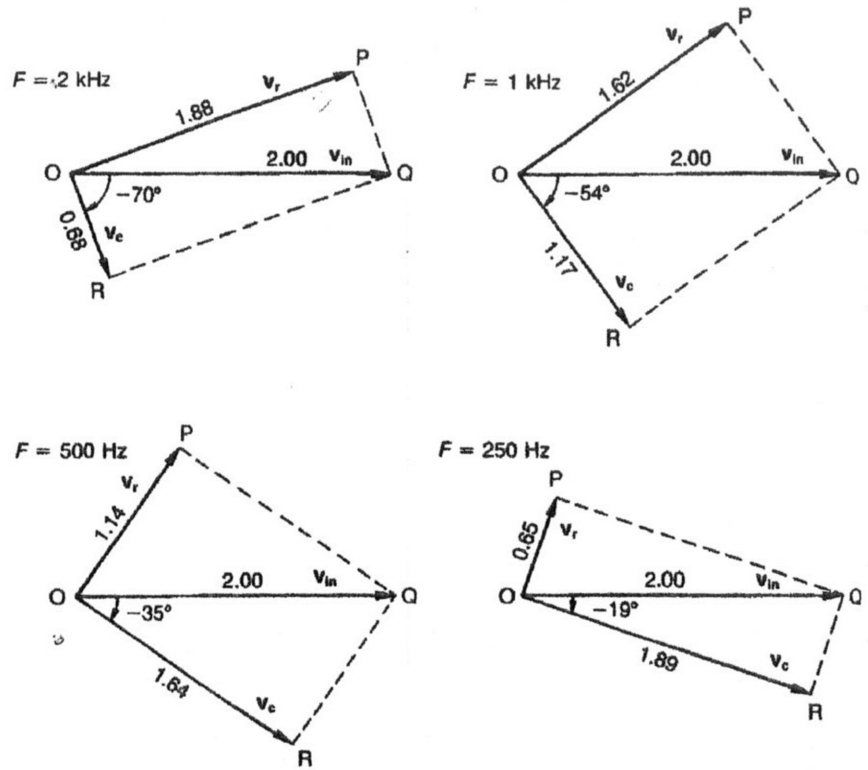

Figure 5.5 *Phasor diagrams for different frequencies show that the $\mathbf{v_c}$ phasor lags less and becomes longer as frequency decreases. This is the characteristic of a low-pass filter*

Keeping up?

3. A high-pass filter can be made by interchanging the resistor and capacitor of Fig. 5.1. Derive a general equation for the output $\mathbf{v_{out}}$ of such a high-pass filter when the input signal is $\mathbf{v_{in}} = V_0 \sin \omega t$.
4. In question 3, to what limits does $\mathbf{v_{out}}$ tend as ω approaches (a) zero (DC) (b) infinity?

Output signals

Figure 5.6 graphs the output signals illustrated by the phasor diagrams in Fig. 5.5. The graph is plotted from 4 ms onward at which time the input signal is at the beginning of a cycle. The output signals all lag behind the input and do not begin their cycles until a fraction of a millisecond later. The graph clearly shows that the lower the frequency, the greater the amplitude of the output signal.

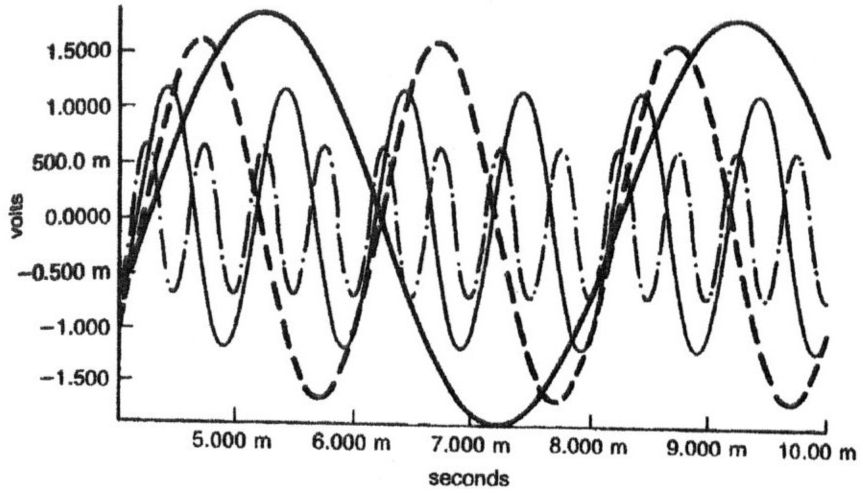

Figure 5.6 *These are the output signals corresponding to the phasor diagrams in Fig. 5.5. The higher the frequency the lower the amplitude*

Transfer function

For a given circuit, its transfer function expresses the relationship between the input and output signals. For a filter circuit of the type shown in Fig. 5.1, the transfer function is:

$$\frac{\mathbf{v_{out}}}{\mathbf{v_{in}}} = \frac{1}{j\omega RC + 1}$$

Remember that the terms $\mathbf{v_{out}}$ and $\mathbf{v_{in}}$ are not just simple voltages. Each represents a sinusoidal voltage with a given frequency, amplitude and phase angle. They have the same frequency ($f = \omega/2\pi$) and the transfer function expresses the relationship between their amplitudes and phase angles. The reason that a single function is able to express the relationship between two different quantities (amplitude and phase) is that the transfer function contains a complex term (in this example, $j\omega RC + 1$), which represents a phasor, and which in its turn represents both amplitude and phase.

Bode plot

The relationship between frequency and output amplitude can be plotted as a graph (Fig. 5.7). The range of Fig. 5.7 is from 0 Hz (DC) to 3 kHz. At 0 Hz, the output amplitude is 2 V, equal to the input amplitude. As frequency increases, amplitude falls steadily, until it reaches about 280 mV at 3 kHz. Amplitudes at 250 Hz, 500 Hz, 1 kHz and 2000 Hz are the same as are drawn in Fig. 5.5. The graph also shows phase lag falling from 0° at 0 Hz to about −78° at 3 kHz. In Fig. 5.7, amplitude is plotted against linear voltage and frequency scales. More often we use logarithmic scales, as in Fig. 5.8, such a graph being known as a **Bode plot**. The values plotted are as before but the shape of the curve is changed by plotting it logarithmically. A logarithmic frequency scale is often used because it is good for displaying the effects of relative frequency changes (for example, doubling or halving frequency) over a wide frequency range.

The pd scaling is not only logarithmic but is expressed in a different unit, the **decibel**. The scale runs from 0 dB at the top where 0 dB corresponds to 2 V, taken as the reference pd for this plot. The amplitude at other frequencies is measured relative to the reference level, in decibels. For example, at 1 kHz, the graph shows that $\mathbf{v}_{out}$ is approximately −4.7 on the decibel scale. From the values quoted in Fig. 5.7, where the amplitude of $\mathbf{v}_{out} = 1.17$ V:

$$20\log(2/1.17) = -4.65\,\text{dB}$$

As with the frequency scale, the logarithmic plot of pd in decibels emphasizes relative values, which are usually more important than absolute ones.

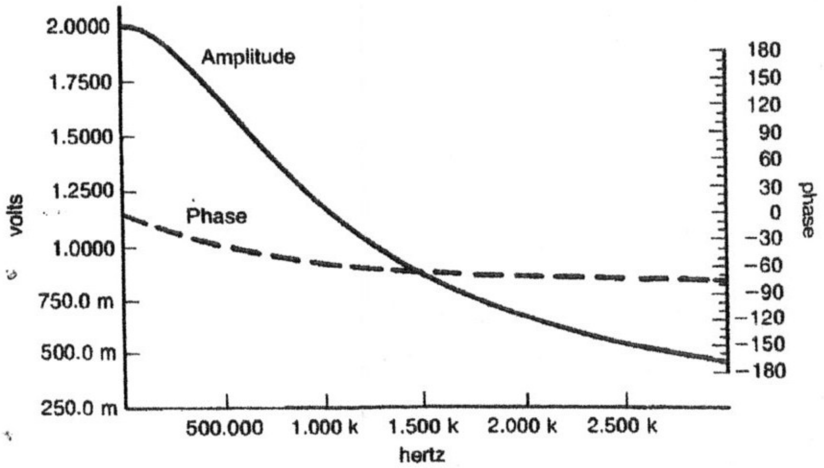

Figure 5.7 *These curves show how the low-pass filter responds to a range of frequencies from DC (0 Hz) to 3 kHz*

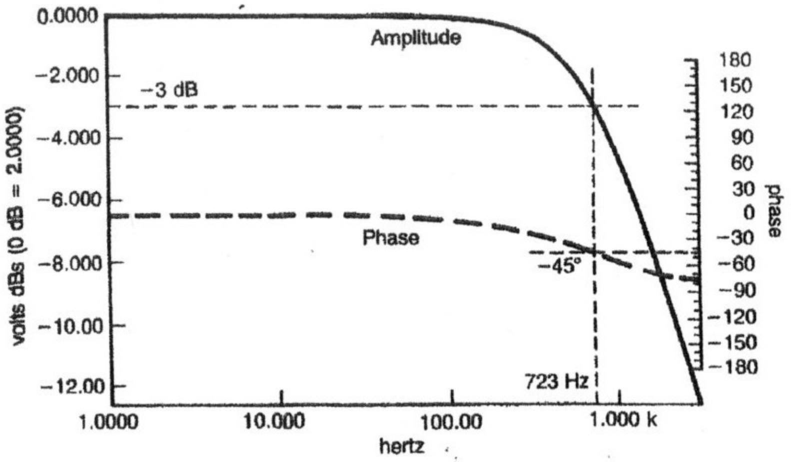

Figure 5.8 *This Bode plot has a logarithmic frequency scale and a decibel output pd scale. At the −3 dB point the frequency is 723 Hz and the phase angle is −45°*

Decibels

Decibels are a way of expressing the ratio between two quantities, x_1 and x_2. If n is the ratio in decibels:

$$n = 10 \times \log_{10}(x_2/x_1)$$

However, in the case of filters and other electronic circuits, as well as in some other applications, the most important consideration is the ratio of powers. Power is proportional to the square of pds or currents. Given pds or currents, the formula for the ratio of powers is:

$$n = 10 \times \log_{10}(v_2^2/v_1^2)$$

The logarithm of a squared number is obtained by doubling the logarithm of the unsquared number, so the easiest formula to use for power ratios is:

$$n = 20 \times \log_{10}(v_2/v_1)$$

In filters, v_1 is the amplitude of the input signal and v_2 is the amplitude of the output signal. There are 3 cases:

Power gain	$v_2 > v_1$	n is positive
Power equality	$v_2 = v_1$	$n = 0$
Power loss	$v_2 < v_1$	n is negative

One way of producing a Bode plot is to take measurements on an actual circuit. A sinusoidal signal of given amplitude, frequency and phase is applied to the input and the amplitude and phase of the sinusoidal output signal are measured. The frequency remains unchanged. Measurements are made at various frequencies, from which the Bode plot is drawn. As we shall see later (Chapter 7) it is also possible to construct a Bode plot from calculated values, given the transfer function of the circuit.

Keeping up?

5. Find the power loss in decibels when (a) $v_1 = 3.5$ and $v_2 = 2.5$, (b) $v_1 = 5$ and $v_2 = 0.5$, (c) $v_1 = 1.2$ and $v_2 = 1.1$.
6. The input to an amplifier has amplitude 0.2 V. The output has amplitude 5.6 V. What is the power gain, in decibels?
7. The open-loop gain of an operational amplifier is said to be 106 dB. If the amplitude of the input signal is 2.5 μV, what is the amplitude of the output signal?

Filter characteristics

The Bode plot of Fig. 5.8 illustrates the main features of a low-pass filter. Towards the left side of the diagram $\mathbf{v}_{out}$ is almost equal to $\mathbf{v}_{in}$. This is the pass band. Toward the right, and on toward even higher frequencies, $\mathbf{v}_{out}$ is very much smaller than $\mathbf{v}_{in}$. This is the stop band. Between the pass band and the stop band is the transition region, in which $\mathbf{v}_{out}$ falls steeply with increasing frequency.

By definition, the pass band of a low-pass filter extends from 0 Hz up to a frequency at which the power of the signal is half that of v_{in}. If the power is half, then $v_{out}^2/v_{in}^2 = 0.5$ and:

$$n = 10\log_{10} 0.5 = 10 \times -0.3010 = -3.010\,\text{dB}$$

This level is marked in Fig. 5.8 as the '−3 dB' line. It is usually referred to as the −3 dB point, the cut-off point, the half-power point, or sometimes as f_C. The graph shows that for this filter, the −3 dB point is at 723 Hz. Phase angle is 0° at 1 Hz, falling through −45° at the −3 dB point and eventually reaching −90° at high frequencies, beyond the right-hand edge of the graph. Figure 5.9 shows the corresponding phasor diagram at the −3 dB frequency, for comparison with those in Fig. 5.5. By definition of the −3 dB point, $\mathbf{v}_{out}^2/\mathbf{v}_{in}^2 = 0.5$. In the phasor diagram, Pythagoras theorem tells us that:

$$\mathbf{v}_{in}^2 = \mathbf{v}_{out}^2 + \mathbf{v}_r^2$$

Dividing by $\mathbf{v}_{in}^2$:

$$1 = \mathbf{v}_{out}^2/\mathbf{v}_{in}^2 + \mathbf{v}_r^2/\mathbf{v}_{in}^2$$

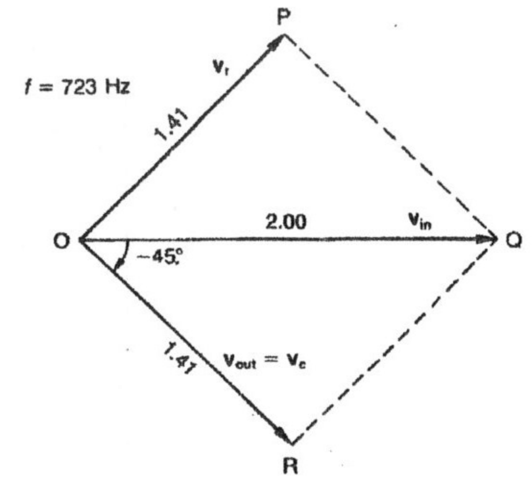

Figure 5.9 *Another phasor diagram to include with those in Fig. 5.5. At 723 Hz, $\mathbf{v}_r = \mathbf{v}_c$, the rectangle becomes a square, the phase angle is −45°. This is when the output power is 3 dB below input level*

Substituting the value for $\mathbf{v}_{out}^2/\mathbf{v}_{in}^2$ at the −3 dB point:

$$1 = 0.5 + \mathbf{v}_r^2/\mathbf{v}_{in}^2$$

$$\mathbf{v}_r^2/\mathbf{v}_{in}^2 = 0.5$$

But it has already been stated that:

$$\mathbf{v}_{out}^2/\mathbf{v}_{in}^2 = 0.5$$

Therefore

$$\mathbf{v}_r^2/\mathbf{v}_{in}^2 = \mathbf{v}_{out}^2/\mathbf{v}_{in}^2$$

$$\mathbf{v}_r^2 = \mathbf{v}_{out}^2$$

$$\mathbf{v}_r = \mathbf{v}_{out}$$

At the −3 dB frequency, the pds across the resistor and capacitor have equal amplitude. This means that their impedances are equal. So the −3 dB is not just a conveniently selected point. It is the frequency at which the resistor and capacitor have equal impedance. In Fig. 5.9, the lengths of the two phasors are equal, and $\mathbf{v}_{out}$ lags 45° behind $\mathbf{v}_{in}$. Ignoring their phase angles, the equality of the sizes of the impedances means that:

$$R = X_c = 1/\omega C$$

Rearranging:

$$\omega = 1/RC$$

But $\omega = 2\pi f$

Rearranging $f = \omega/2\pi$

Substituting $f = 1/2\pi RC$

This equation is used for calculating the −3 dB frequency, given the values of R and C. For example, in Fig. 5.8, $R = 220\,\Omega$, $C = 1\,\mu F$ and:

$$f = \frac{1}{2\pi \times 220 \times 10^{-6}} = 723\,\text{Hz}$$

This agrees with the value of the −3 dB frequency obtained from the graph.

Cut-off frequency

At the cut-off frequency, or −3 dB point, of a resistor/capacitor low-pass filter:

Impedances of resistor and capacitor are equal in size.
Phase angle is −45°
$f = 1/2\pi RC$

Roll-off

The rate at which amplitude falls off in the transition region is an important characteristic of a filter. The steeper the slope of the curve, the more sharply does the filter distinguish between those frequencies it passes at almost full amplitude and those that it strongly blocks. The equation on p. 71 shows that:

$$\mathbf{v}_{out} = \frac{\mathbf{v}_{in}}{j\omega RC + 1}$$

If frequency is reasonably high, as it is above the cut-off point, we can ignore the 1 in the denominator and, since we are not concerned with phase, we can ignore j too:

$$\mathbf{v}_{out} = \frac{\mathbf{v}_{in}}{\omega RC}$$

$\mathbf{v}_{out}$ is inversely proportional to ω. If frequency is doubled, the amplitude of $\mathbf{v}_{out}$ is halved. A doubling of frequency is often referred to as an octave, a word borrowed from musical terminology. Hence, the ratio of two outputs for two signals an octave apart is 0.5. In decibels, this is $20 \log 0.5 = -6$ dB. For a

doubling of frequency, the output falls by 6 dB. This rate of roll-off is the general rule in most simple filters.

Filter action

The action of a filter on a single sine wave is to reduce its amplitude and cause a phase lag. If we filter a more complicated signal, such as a sawtooth wave (p. 28), each sinusoidal component of the signal is affected in amplitude and phase depending on its frequency. Figure 5.10 shows a sawtooth wave before and after passing through a low-pass filter. Like the signal of Fig. 2.8(d), the wave has a frequency of 0.159 Hz and an amplitude of 1.58 V. Because this is a low-frequency signal, we are using a filter with a low −3 dB point. The capacitor is 1 μF as before but the resistor is increased to 390 kΩ. The −3 dB point is:

$$f_c = 1/2\pi RC = 1/(2\pi \times 390\,000 \times 10^{-6}) \approx 0.4\,\text{Hz}$$

The effect of the filter on the triangular wave is visually obvious. The sharp downward-pointing corners of the signal become rounded off. Some of its high-frequency components are being lost or, at least, reduced in amplitude.

The filter action is investigated further by a Fourier analysis (Fig. 5.11). This has a similar appearance to the analysis of the original unfiltered signal (Fig. 2.12). But both figures have been plotted with the same vertical scale to make it clear that the amplitudes of the fundamental and harmonics are reduced

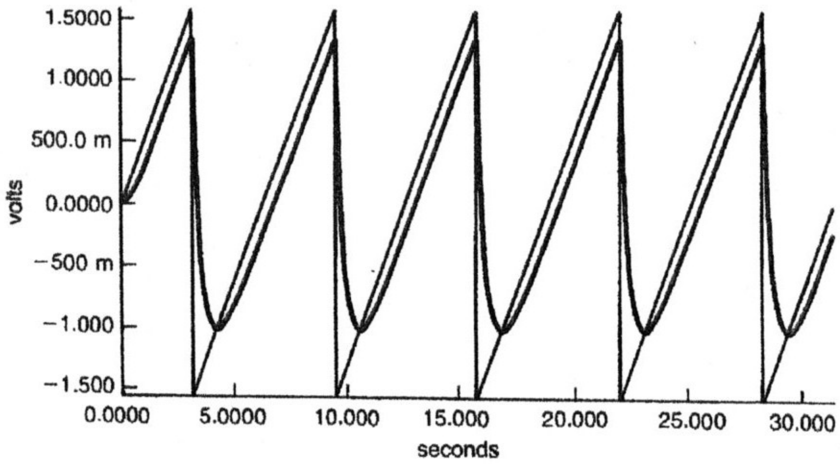

Figure 5.10 *The effect of a low-pass filter on a sawtooth signal is to reduce its amplitude, round off some of the corners and introduce a phase delay*

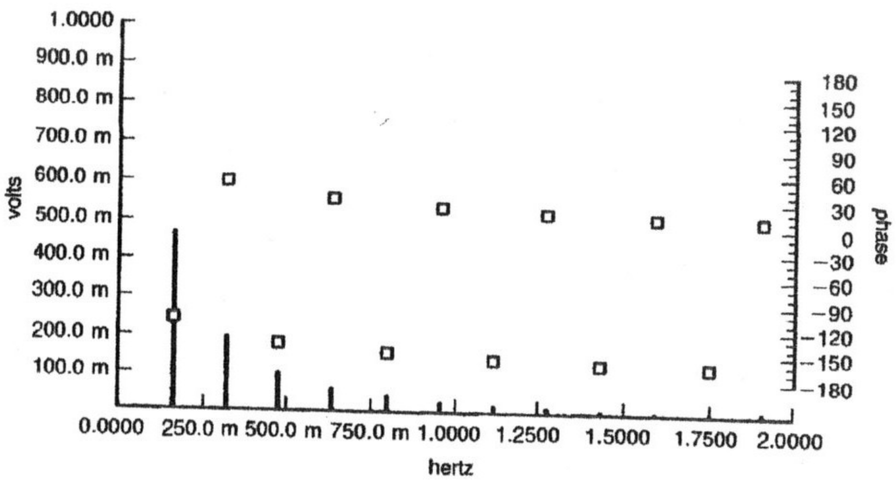

Figure 5.11 *A Fourier analysis reveals the relatively greater attenuation of high-frequency components after filtering a sawtooth signal through a low-pass filter (compare with Fig. 2.12). Phase lag increases with increasing frequency*

by filtering. The effect is least for the fundamental and the first few harmonics. Measurements on the graphs confirm this effect:

Frequency	*Amplitudes* (mV) v_{in} *(unfiltered)*	v_{out} *(filtered)*	v_{out}/v_{in}
Fundamental	1000	465	0.465
1st harmonic	500	196	0.392
2nd harmonic	333	108	0.324
3rd harmonic	250	69	0.276
4th harmonic	200	45	0.225
5th harmonic	167	30	0.180
6th harmonic	143	24	0.168

The dashed line in the table shows where the cut-off frequency is located. The last column of the table shows that attenuation of amplitude increases progressively with increase in frequency. But, as illustrated in Fig. 5.7, this is a gradual effect. There is no sharp edge to the pass band. Ideally the pass band is flat and a sudden and steep roll-off begins at the −3 dB frequency. To obtain this we need a filter of more elaborate design.

Figure 5.11 shows a further effect of filtering. In the original signal (Fig. 2.12) the phase of each component is either +90° or −90°. In the filtered signal there is a gradual increase in phase lag as frequency increases. This is in accordance with the effect of frequency on filter characteristics. The result is that the components of a signal are each delayed by a different amount. This is an additional source

of distortion in the filtered signal for, not only are the harmonics attenuated by different amounts, but they arrive at the output of the filter at slightly different stages in their cycles. This effect is known as group delay. It is particularly important with high-frequency pulsed signals, such as are often found in digital circuits.

High-pass filters

The action of high-pass filters has already been the subject of some of the questions in Keeping up? We found that the transfer function of a resistor/capacitor high-pass passive filter is:

$$\frac{v_{out}}{v_{in}} = \frac{j\omega RC}{j\omega RC + 1}$$

Once the action of a low-pass filter has been studied and understood, there is little new to be learned about high-pass filters. It is easy to convert one sort into the other. In the case of a passive low-pass resistor/capacitor filter, simply exchanging the resistor for the capacitor turns it into a high-pass filter. The Bode plot for a high-pass filter is similar to that for a low-pass filter but reversed from left to right. Compare Fig. 5.12 with Fig. 5.8, both of which are based on the same values of R and C. In Fig. 5.12 the frequency range has been extended up to 100 kHz to allow more of the pass band to be plotted. Another difference in the plots is that the decibel scale extends down to −56.58 dB to cover the curve down to 1 Hz. This represents an attenuation of ×0.06.

The transition region has a slope of +6 dB per octave and reaches the −3 dB point at 723 Hz, the same frequency as in a low-pass filter. In the pass band, above

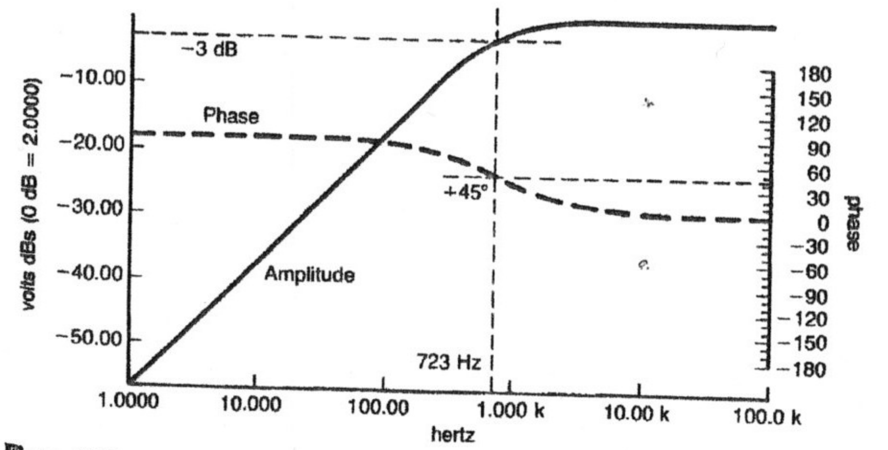

Figure 5.12 *Compare these frequency response curves of a high-pass filter with those in Fig. 5.8*

about 10 kHz, the filter transmits the signal with virtually no loss. Changes of phase are the opposite to those found in a low-pass filter because phase angles are positive instead of negative. Phase angle is +90° at 1 Hz, falling to exactly +45° at the −3 dB frequency. As frequency increases beyond that, the phase angle gradually levels out to 0°.

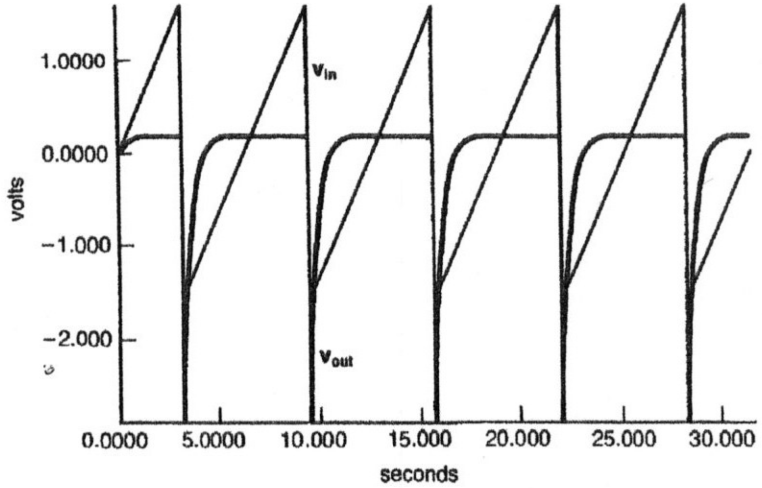

Figure 5.13 *The effect of filtering a sawtooth signal through a high-pass filter*

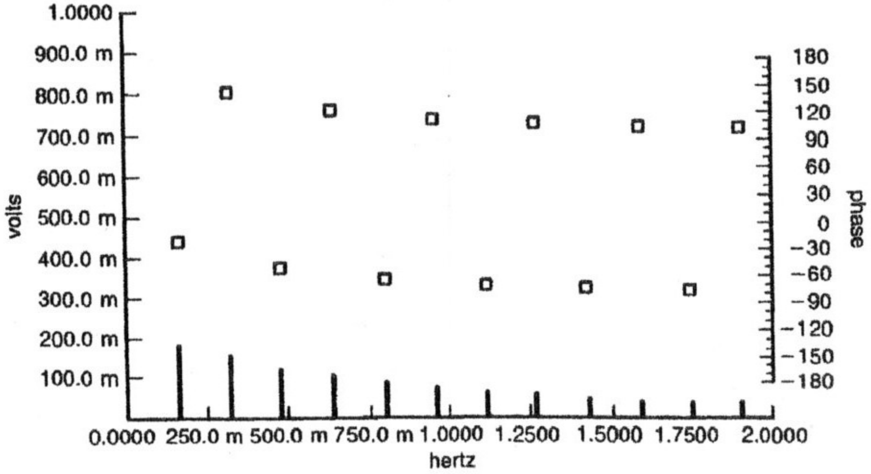

Figure 5.14 *The frequency spectrum loses much of its lower-frequency components after high-pass filtering. Compare this spectrum with Figs 2.12 and 5.11*

The effect of the high-pass filter on a sawtooth wave is a complete distortion of its shape (Fig. 5.13) due to the reduction in amplitude of the fundamental and lower harmonics. In the frequency spectrum (Fig. 5.14), the fundamental is reduced to about 0.18 of its value in an unfiltered signal (Fig. 2.12), and there is a general 'flattening' of the spectrum, so that the higher harmonics become relatively more significant and the waveform exhibits more pronounced 'spikes'.

Inductive filters

The filtering action of resistor/capacitor filters depends upon the way the impedance of the capacitor varies with frequency. The impedance of a capacitor is inversely proportional to frequency, in other words, increasing frequency leads to reducing impedance. Depending on the arrangement of the resistor and capacitor, we obtain a low-pass or high-pass filter (Fig. 5.15(a) and (b)).

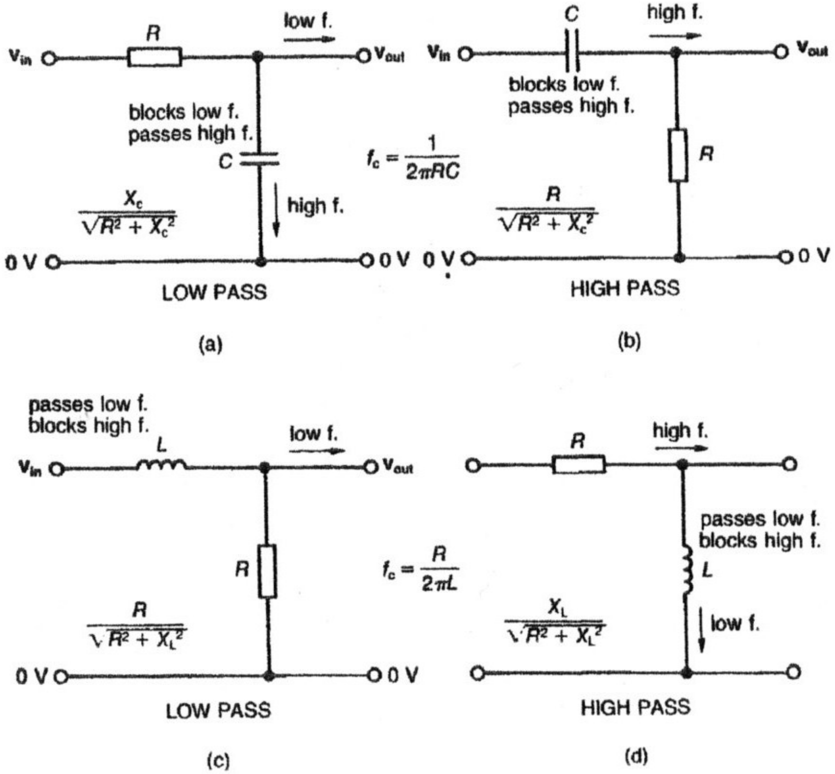

Figure 5.15 *A summary of the four types of first-order passive filters. The formula at the bottom left of each diagram gives the transfer ratio v_{out}/v_{in} at any given frequency*

An inductor is also frequency dependent, but its impedance is directly proportional to frequency. Increasing frequency leads to increasing impedance. This property can be used in building a low-pass or high-pass filter from a resistor and an inductor (Fig. 5.15(c) and (d)). The main reason why inductors are seldom used in practical filters is that filtering frequencies in the audio range and below requires the inductor to be unduly large and heavy. This goes against the present-day trend toward light, small and portable equipment. Another reason for the unpopularity of inductors is that they generate magnetic fields which may interfere with nearby circuits. Conversely, they may also pick up magnetic interference unless they are very thoroughly shielded. It is only in high-frequency circuits, such as high-frequency radio or microwave circuits, that inductors can be small enough to be practicable.

Figure 5.15 summarizes the structure and properties of filters in which there is just one reactance, either a capacitor or an inductor. These are known as first-order filters. The figure includes the formulae for f_c and for the ratio v_{out}/v_{in} for each type of filter. These ratios may be deduced from the geometry of phase diagrams like those of Figs 5.5 and 5.9.

Test yourself

1. Design a passive low-pass resistor/capacitor filter with a −3 dB point of 500 Hz, using a 10 kΩ resistor. Calculate the attenuation of a 600 Hz sine wave signal when passed through this filter.
2. Design a passive high-pass resistor/capacitor filter with a −3 dB point of 2 kHz, using a 2.2 nF capacitor. Calculate the attenuation of a 1 kHz sine wave signal when passed through this filter.
3. Design a passive high-pass resistor/inductor filter with a −3 dB point of 10 MHz, using a 10 kΩ resistor. Calculate the attenuation of a 9 MHz sine wave signal when passed through this filter.

6 Second-order passive filters

The filters described in Chapter 5 each have a single reactive component, either a capacitor or an inductor. Because of this, they are known as **first-order filters.** Although a first-order filter is adequate for many purposes, it has two important drawbacks:

1. The pass-band merges with the transition region, so that there is no sharp differentiation between frequencies that are to be passed and those that are to be attenuated.
2. The slope of the response curve in the transition region is only −6 dB per octave, with the result that frequencies several octaves away from the nominal cut-off point are present in the output from the filter.

Improved filtering is obtained by using a **second-order filter,** which contains two reactive components. The 'knee' of the response curve between the pass-band and the transition region can be made sharper in such a filter. A second-order filter usually has a steeper response curve in the transition region. A further possibility in a second-order filter is to combine the low-pass function with the high-pass function to produce band-pass and band-stop filters, as described later in this chapter.

Two-capacitor filters

Two low-pass resistor/capacitor filters may be connected **in cascade** (Fig. 6.1). The output from the first filter is fed to the second filter. Suppose that $R1 = R2 = 220\,\Omega$ and $C1 = C2 = 1\,\mu$, making both stages the same as the circuit analysed in Fig. 5.2. We also use the same input signal, $v = 2 \sin 2000\pi t$. Figure 6.2 plots the result of filtering. In order of decreasing amplitude, the curves are the original signal (v_{in}), the output of the first stage (v_1), and the output of the second stage (v_{out}). Although v_{in} has an amplitude of 2 V, the same as that in Fig. 5.2, the output of the first stage v_1 of Fig. 6.2 is only 0.81 V contrasted with 1.17 V for v_{out} of Fig. 5.2. This is because the first stage of the two-stage filter has had to supply current to the second stage, causing a drop in pd across $C1$. There

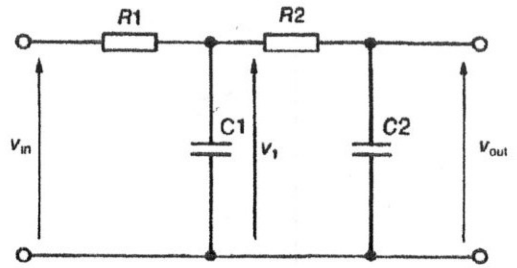

Figure 6.1 *Cascading two low-pass filters increases roll-off but reduces output amplitude*

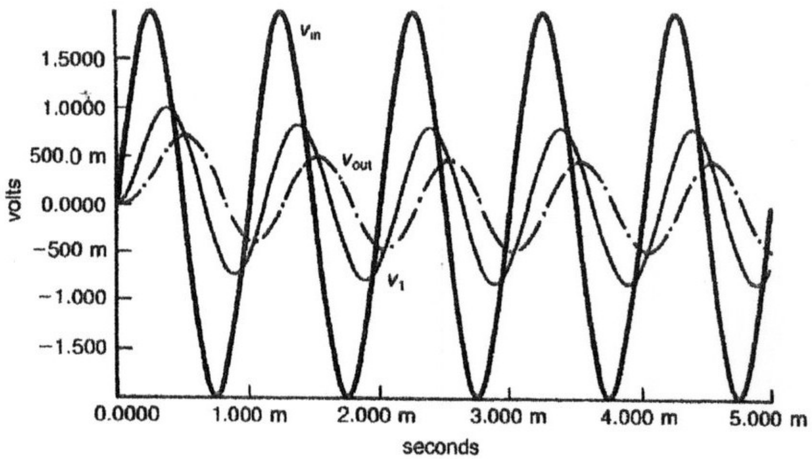

Figure 6.2 *In a cascaded low-pass filter (Fig. 6.1) the amplitude is reduced at each stage and the phase angle is increased*

is a further drop in amplitude in the second stage so that v_{out} for Fig. 6.2 is only 0.47 V. Overall, the second-order filter causes amplitude to fall from 2 V to 0.47 V. This cascading effect makes the behaviour of the filter depart from the theoretical predictions, which assume that the output of a filter stage is fed to a high impedance input at the next stage. This is something that cannot be done with passive components.

Looking at phase changes, the first stage of the second-order filter produces a phase lag of 51°. This is slightly less than that produced by the first-order filter. But the overall effect of the second-order filter is a lag of 106°. Summing up, the second-order filter has greater attenuation and bigger phase lag at 1 kHz than the first-order filter. But the action of the filter at any particular frequency is not of primary interest. The essential point about a filter is that it operates over a wide range of frequencies, and we should study its action over an appropriate range.

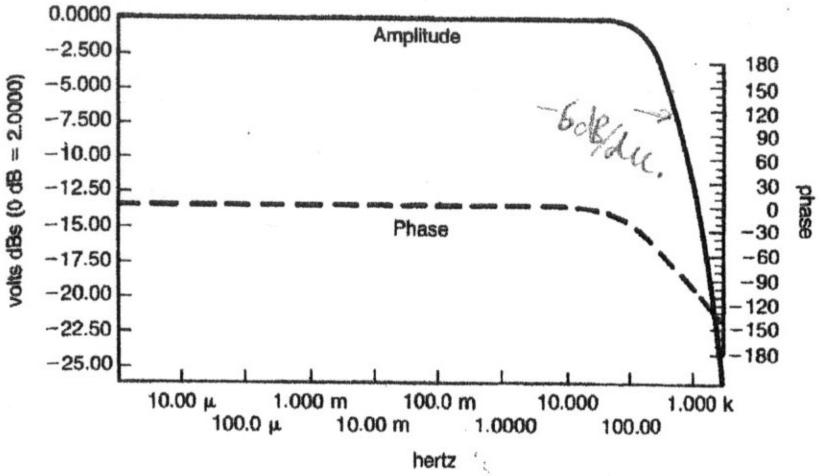

Figure 6.3 *A cascaded low-pass filter has a roll-off of −12 dB per octave*

To do this, we use a Bode plot (Fig. 6.3). This demonstrates that the second-order filter is an improvement on the first-order filter in that the rate of fall in the transition region is doubled to −12 dB per octave or slightly more, due to the cascading effect. This effect also causes the −3 dB point to occur at a lower frequency.

When the resistors and capacitors of Fig. 6.1 are exchanged, the filter becomes a second-order high-pass filter. Again there is greater attenuation than in a first-order filter but the cut-off is sharper. It is also possible to cascade resistor/inductor filters with similar results.

Capacitor/Inductor filters

When a capacitor and an inductor are present in the same filter, as in Fig. 6.4, there are two reactive devices responding in opposite ways to changes of

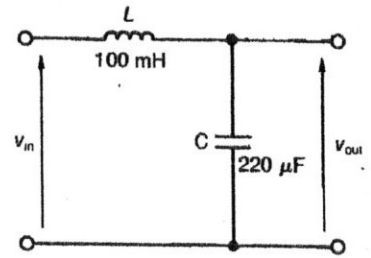

Figure 6.4 *This low-pass filter comprises two devices which have opposite reactions to frequency*

frequency. The inductor blocks high frequencies and passes low frequencies. The capacitor passes high frequencies to the 0 V line but blocks low frequencies. Their combined effect in this circuit is to produce a low-pass filter with enhanced action. The Bode plot produces some surprises (Fig. 6.5). Far from there being a gentle 'knee' on the frequency response curve, there is a very sharp spike at 33.9 Hz. To gain a full appreciation of this, we replot the response with actual voltages on the *y*-axis (Fig. 6.6). Output amplitude remains close to input amplitude (2 V) at low frequencies, but rapidly rises to 181 V at 33.9 Hz. Then it falls very steeply with increasing frequency. Phase shows similarly striking behaviour. At low frequencies, the phase angle is 0°, but at 33.9 Hz, it swings very rapidly to −180°.

A clue to this behaviour may be understood by calculating the reactances at 33.9 Hz. First of all, $\omega = 2\pi f = 213$ rad/s. For the capacitor:

$$X_C = -j/(213 \times 220 \times 10^{-6}) = -j21.3$$

For the inductor:

$$X_L = j \times 213 \times 100 \times 10^{-3} = j21.3$$

When the frequency is 33.9 Hz, the inductances are equal in magnitude, but opposite in direction. Because of the sign of j, pds across them are exactly 180° out of phase. At any instant the pds are of opposite signs and the total pd across the capacitor and inductor is zero. In this state, the circuit is **resonant**. The frequency at which this occurs is known as the resonant frequency.

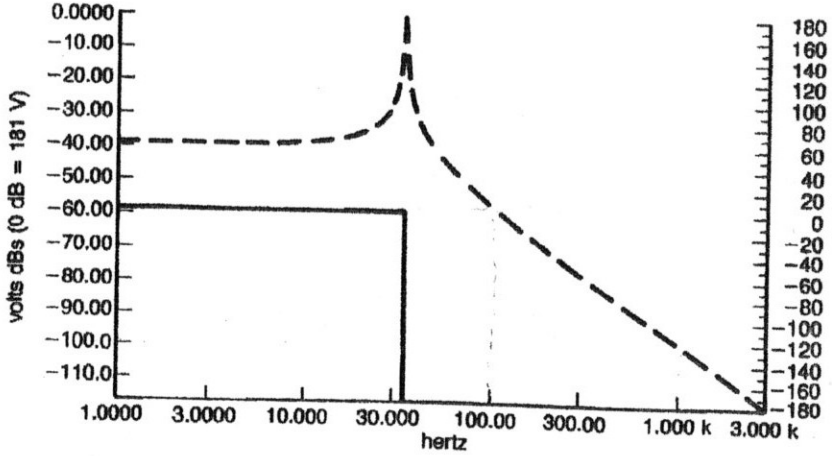

Figure 6.5 *The filter of Fig. 6.4 has a dramatic response to frequency, both in the amplitude and the phase of the output signal*

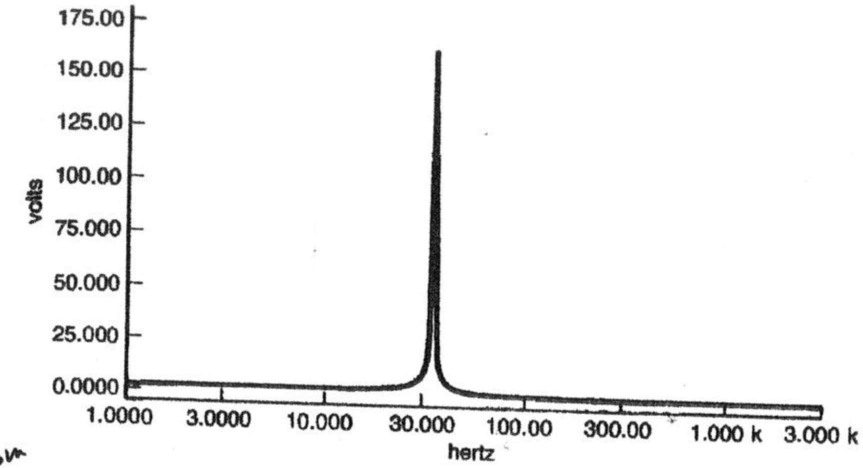

Figure 6.6 *Plotting Fig. 6.5 on linear scales makes the filter's amplitude response seem even more impressive*

Keeping up?

1. In a capacitor/inductor filter there is a peak in the response at 500 Hz. What can we say about the capacitor and inductor?
2. In what ways does the frequency response of two cascaded first-order low-pass filters differ from that of a first-order filter?

Resonance

The effect is like that of pushing a child on a swing. A swing, being a pendulum, has a natural frequency of swinging. Even if we push the swing gently every time the child is swinging away from us, we supply a little energy to the swing at each push. This energy, though small, is greater than that lost from the swing by air resistance and friction. Gradually the amount of energy in the system increases and the child swings higher and higher. The swing resonates at its natural frequency. If we push at a different rate, there are times when we are adding energy to the swing, but there are also times when we push the swing as it is coming towards us. Then we are reducing the energy of the swing and it swings less high; there is no resonance.

A similar effect is noticed in a room, such as a bathroom, when the walls, floor, ceilings and furnishings are reflective of sound. There are several frequencies at which the air of the room vibrates. If we sing a note at one of these frequencies, we supply energy to the air in phase with its natural vibrations. The amplitude of the vibrations builds up and the air 'booms' loudly at that frequency.

In the filter, a small amount of energy supplied from the input source at each cycle soon builds up to a large oscillating pd.

At resonance we have:

$$X_C = X_L$$

or

$$\frac{1}{\omega C} = \omega L$$

Here we are concerned only with the magnitudes of the signals, so we can omit j. Rearranging this equation:

$$\omega^2 = \frac{1}{LC}$$

$$\Rightarrow \quad \omega = \frac{1}{\sqrt{(LC)}}$$

$$\Rightarrow \quad f = \frac{1}{2\pi\sqrt{(LC)}}$$

Substituting $C = 220\,\mu F$ and $L = 100\,mH$ into this equation yields the result $f = 33.9\,Hz$, which confirms the value in Fig. 6.5. At this frequency the current drawn from the source is limited only by the output impedance of the source, the resistances of the inductor coil and the leads of the capacitor. Such impedance is likely to be low; call it R. The current to the capacitor is $i = v_{in}/R$. The current i is the same through both capacitor and inductor and we are interested in the pd across the capacitor, for that is where we are acquiring the output from the filter. If the current through the capacitor is i and its impedance at resonance is X_{Cr}, the pd across it is:

$$\mathbf{v_{out}} = iX_{Cr} = \mathbf{v_{in}}/R \cdot X_{Cr} = \mathbf{v_{in}}/(\omega_r CR)$$

where ω_r is the value of ω at the resonant frequency. The gain of the filter is:

$$\frac{\mathbf{v_{out}}}{\mathbf{v_{in}}} = \frac{1}{\omega_r CR}$$

At the resonant frequency and for a given value of C, the gain of the filter is inversely proportional to R. If R is small enough, the gain is more than 1. In other words, $\mathbf{v_{out}}$ is greater than $\mathbf{v_{in}}$. This is why it is possible to obtain an output amplitude of 181 V when the input amplitude is only 2 V, as in Fig. 6.6. Remember that Fig. 6.6 shows only the amplitude of the pd signal across the capacitor. At any instant the pd across the inductor is equal and opposite to the pd across the capacitor, so this large pd never appears across the pd generator.

The combination of capacitor and inductor has introduced a new feature into the filter. This is the possibility of producing resonance at a required frequency and so sharpening the knee of the frequency response curve. In Fig. 6.6 the knee is far too sharp to make a satisfactory filter but, by increasing the value of R, we can fashion the knee to the shape we require. Since ω_r is fixed by the value of L and C and since C itself is fixed, the gain can be controlled by selecting a suitable value for a resistor placed in series with the inductor (Fig. 6.7). In Fig. 6.8 the curves show the frequency response with resistors of different values. From top to bottom, the values of R are 0, 20, 40, 60, 80 and 100 ohms. The resistor introduces damping into the system. There is virtually no damping when $R = 0\,\Omega$, and we say that the filter is **underdamped**. When

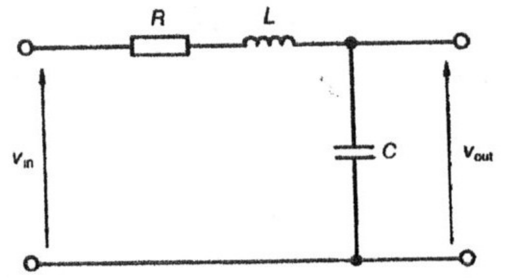

Figure 6.7 *Adding a resistor to the inductor/capacitor filter of Fig. 6.4 allows critical damping to be achieved*

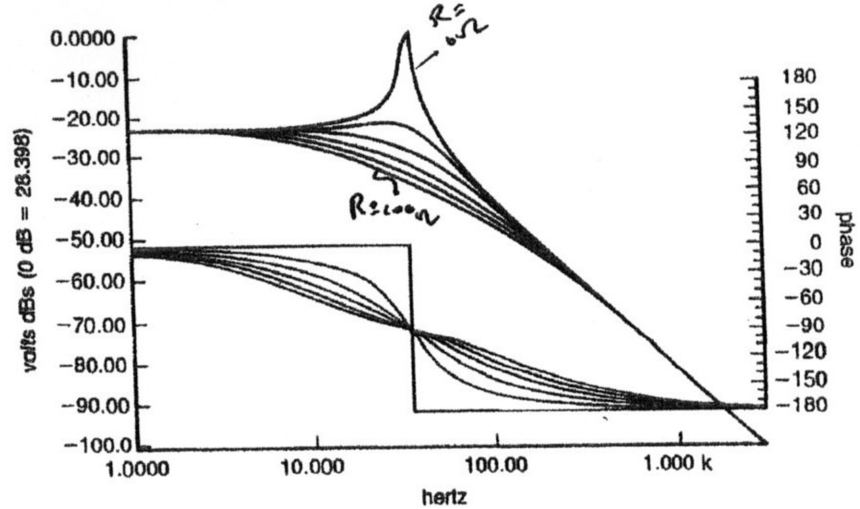

Figure 6.8 *These curves illustrate the effects of sweeping the damping resistor of Fig. 6.7 through a range of values from 0 Ω (top curves) to 100 Ω in steps of 20 Ω*

$R = 100\,\Omega$ the knee of the curve is no more prominent than in an ordinary resistor/capacitor filter (Fig. 5.15(a)) and the beneficial effect of the inductor is lost. The filter is **overdamped**. Somewhere between these extremes there is a value for R which produces **critical damping**. Judging from the figure, it looks as if the filter is slightly underdamped when $R = 20\,\Omega$, and is overdamped when $R = 40\,\Omega$. Further experimenting within this range shows that $R = 30\,\Omega$ results in a curve that remains level in the lower frequencies, then drops sharply, but without a hump. The filter is **critically damped**. Measurement on the transition region of the curve reveals that the −3 dB point is at 34 Hz and that amplitude falls by 12 dB per octave, which is typical of second-order filters.

The combination of capacitor and inductor, together with a resistor for the control of damping, produces a filter with a sharper knee and a more rapid roll-off in the transition region. The same principles can be applied to filters of other designs. We can build high-pass filters, and can cascade several low-pass filters or several high-pass filters for even sharper cut-off and steeper roll-off (though with considerable attenuation). In the case of cascaded filters containing inductors, the chief limitation is the physical size (and often weight) of the inductors required. It is only at high frequencies that capacitor/inductor filters are really practicable (see p. 82).

Keeping up?

3. What is the resonant frequency of a capacitor/inductor low-pass filter in which $C = 22\,\mu F$ and $L = 15\,\mu H$?
4. What is the reason for including a series resistor in a capacitor/inductor filter?

Band-pass filters

The idea of cascading two filters makes it possible to build a band-pass filter. Basically, all we need to do is to cascade a low-pass filter (for example, Fig. 5.15(a)) with a high-pass filter (Fig. 5.15(b)). If the cut-off points are chosen correctly, the combination cuts out low frequencies and high frequencies, leaving a pass-band of intermediate frequencies. There are several ways in which this can be done, with combinations of resistors, capacitors and inductors in various configurations. The same principles apply to all, so we shall study only one, which is similar to the low-pass filter we have just studied. It has an inductor, capacitor and resistor in series as in Fig. 6.7, but now the output is the pd across the resistor (Fig. 6.9). Giving the inductor and capacitor the same values as before, and making $R = 30\,\Omega$ to produce critical damping, the frequency response shows a peak at 33.9 Hz (Fig. 6.10). In a band-pass filter, this is known as the centre frequency. The roll-off is +6 dB per decade on the low-frequency side and −6 dB per decade on the high-frequency side. Second-order low-pass and high-pass filters normally have a roll-off of −12 dB per octave but in a band-pass filter the

R = 30Ω

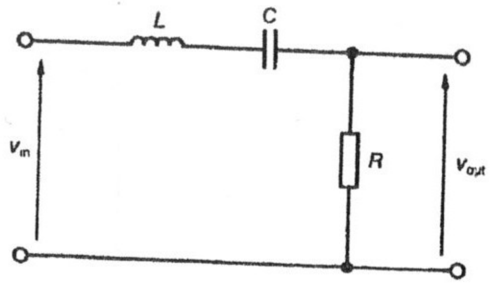

Figure 6.9 *An inductor and capacitor in series produce a band-pass filter. The resistor acts to dampen the response at the resonant frequency*

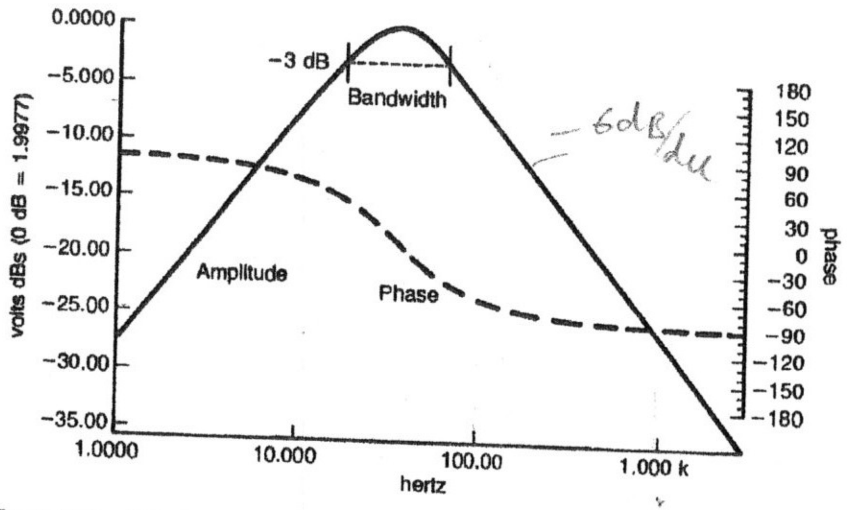

Figure 6.10 *A Bode plot of the action of the band-pass filter of Fig. 6.9. The −3 dB line cuts the amplitude curve at 19.4 Hz and 59.1 Hz, giving a bandwidth of 39.7 Hz*

roll-off is distributed equally on both sides of the centre frequency, −6 dB on each side.

The two important characteristics of a band-pass filter are **bandwidth** and the **selectivity**. By definition, the bandwidth is the difference between the upper and lower frequencies at which the output is 3 dB below the input. The bandwidth gauges the extent of the pass-band. In Fig. 6.10 the −3 dB points are at 59.1 Hz and 19.4 Hz, so the bandwidth is given by:

$$BW = 59.1 - 19.4 = 39.7\,\text{Hz}$$

The size of the bandwidth does not indicate how effective the filter is in any given application. A bandwidth of 39.7 Hz is very narrow if the frequencies we are dealing with are of the order of hundreds of kilohertz. We would rank such performance as highly selective. But with frequencies of only a hundred or so hertz, a pass-band of 39.7 Hz is relatively wide and the filter is considered to be unselective.

The selectivity of the filter, Q, relates bandwidth to the resonant frequency f_r of the filter:

$$Q = \frac{f_r}{\mathrm{BW}}$$

For example, in the filter of Fig. 6.9, BW = 39.7 and $f_r = 33.9$, and:

$$Q = \frac{33.9}{39.7} = 0.85$$

Since Q is a ratio it has no units. Q appears in various guises in electronic circuits, often being referred to as the **quality factor.**

Although the response curve of Fig. 6.10 looks symmetrical, this is because it is plotted on a logarithmic frequency scale. With actual values, the resonant frequency does not lie half-way between the lower and upper −3 dB points. In other words f_r is not the arithmetic mean of the lower and upper points. Instead, it is the geometric mean:

$$f_r = \sqrt{(19.4 \times 59.1)} = 33.9\,\mathrm{Hz}$$

Keeping up?

5. The upper and lower −3 dB points of a band-pass filter are 2420 kHz and 2300 kHz. What are its bandwidth and its selectivity or quality factor?
6. A band-pass filter has a bandwidth of 250 Hz and a Q of 15. What are its resonant frequency and its upper and lower −3 dB points?

Band-stop filter

When a capacitor and inductor are wired in series, they act as a band-pass filter. When wired in parallel (Fig. 6.11) the capacitor passes high frequencies and the inductor passes low frequencies. All frequencies are passed except those which are too low to pass through the capacitor and too high to pass through the inductor. In this way we obtain a filter which passes all frequencies except those within an intermediate band. This is a **band-stop filter**, sometimes known as a **notch filter** if the band is narrow. A typical response is illustrated in Fig. 6.12, using components of the same values as used in the previous band-pass filter. The output amplitude is constant at 2 V over most of the frequency range but plunges sharply to reach a minimum value of −25 dB at the resonant frequency 33.9 Hz. Calculations of bandwidth and Q are the same for this filter as for a band-pass filter.

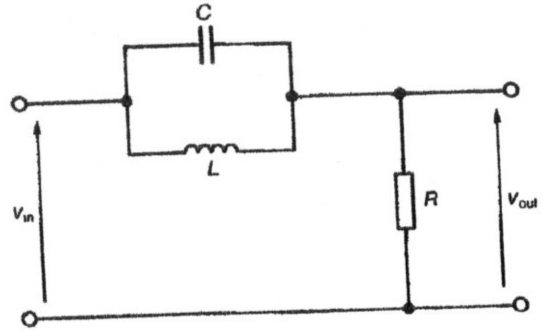

Figure 6.11 *This band-stop filter is based on a capacitor and inductor connected in parallel*

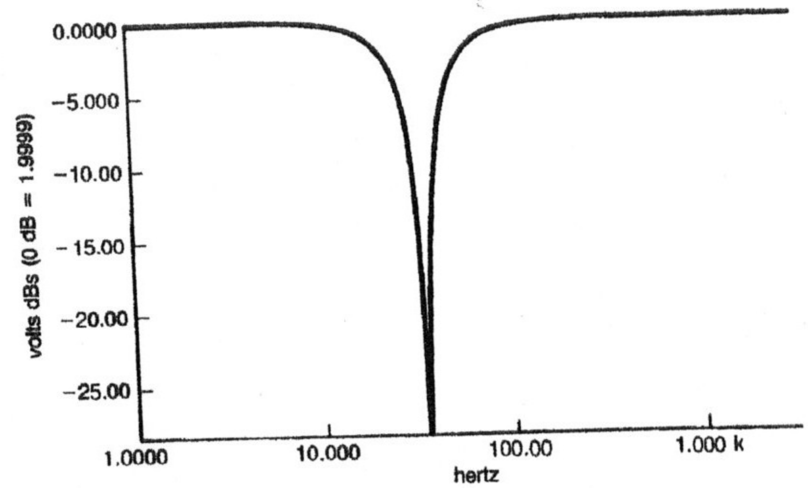

Figure 6.12 *The Bode plot of the filter in Fig. 6.11. If $C = 220\,\mu F$, $L = 100\,\mu H$ and $R = 30\,\Omega$ the curve dips sharply to −25 dB at 33.9 Hz*

The band-stop combination of a capacitor and inductor in series may also be used for band-pass filtering. As an example, take the filter of Fig. 6.13. This has two band-pass sections in cascade. Between these two sections there is a band-stop section. Very high and very low frequencies are passed through this section to the 0 V line, thus enhancing the action of the two band-pass filters. There are many other filter designs based on this principle.

Filters operating at frequencies of a few hundred megahertz may be built from capacitors of a few picofarads and inductors of a few tens of nanohenries. Although their capacitances and inductances are so small, the high frequency

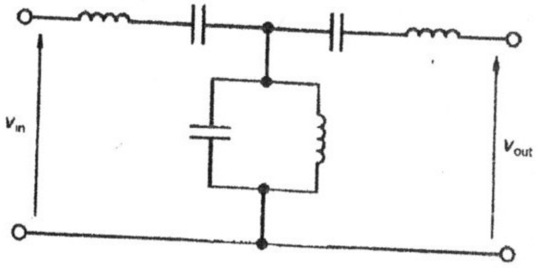

Figure 6.13 *The series-connected capacitors and inductors act as band-pass filters. Their action is enhanced by the parallel-connected capacitor and inductor which constitute a band-stop filter, which prevents intermediate frequencies from passing through to the 0 V line*

results in reasonably high reactances. For example, at 200 MHz, an inductor of 50 nH has a reactance of 63 Ω. In the gigahertz range it is not possible to manufacture capacitors and inductors of suitably low value. Even if it were, the inductance of component leads and stray capacitances in inductors would play an unduly large part in affecting circuit behaviour. Instead of inductors and capacitors, we use a shaped **microstrip**. A microstrip is a strip of conductor coated on a board of insulating material, with a continuous coating of conductor (the ground plane) on the reverse side. It is similar in appearance to one of the tracks on a printed circuit board. Microstrips are used as transmission lines for conveying microwave signals from one part of a circuit to another. If the width of the microstrip varies abruptly, as in Fig. 6.14, the variations in width cause distortions in the electrical field between the microstrip and the ground plane. The interaction between this distorted field and the currents within the microstrip produce effects that are analogous to capacitance and inductance of

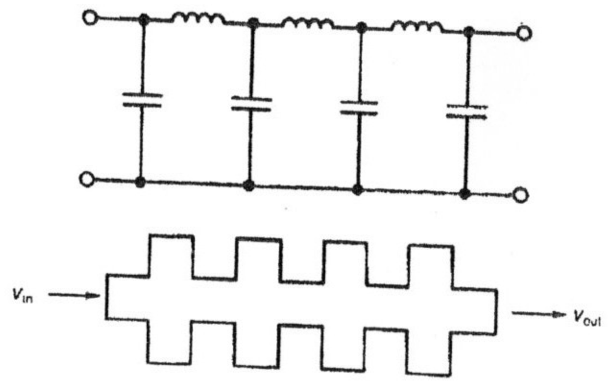

Figure 6.14 *The action of this seventh-order low-pass capacitor/inductor filter can be duplicated at ultra-high frequency by a length of microstrip shaped as shown*

very low magnitude. In Fig. 6.14, the shape of the microstrip produces an effect equivalent to that of a low-pass inductor/capacitor filter. The inductances and capacitances are determined by the dimensions of the microstrip. In this way we can produce filters with the very small inductances and capacitances required for filtering at ultra-high frequencies.

Summary

When a passive filter includes two or more reactive devices, it makes it possible to design band-pass and band-stop filters in addition to low-pass and high-pass filters. A filter that comprises one or more capacitors and one or more inductors is capable of resonance. This effect can be used to sharpen the response of the filter, and a resistor is generally used to dampen the response to its critical level. The performance of a band-pass or band-stop filter is characterized by its bandwidth and its selectivity, or quality factor. The main disadvantage of multi-stage passive filters is that the signal is severely attenuated.

Test yourself

1. Identify the filter types in Fig. 6.15(a) and (b).
2. Given that the capacitors in Fig. 6.15(a) are 47 nF, what inductances are needed to produce resonance at 25 kHz?
3. What roll-off would you expect in the filter of Fig. 6.15(b)?
4. Describe the behaviour of (a) capacitors and (b) inductors when signals of low and high frequency are passed through them.

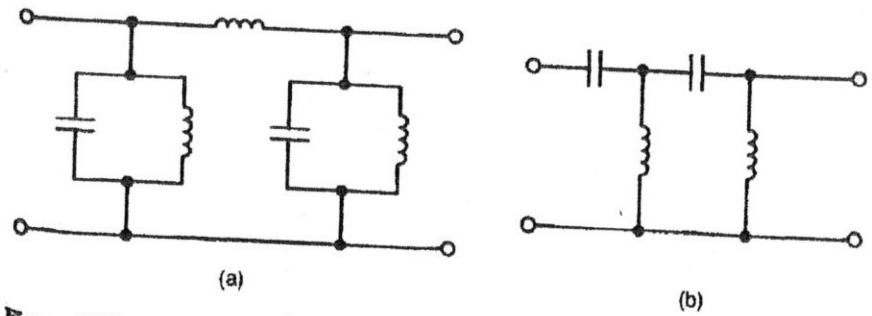

Figure 6.15 *These filter designs are the subjects of question 1*